Ultrametric matrices and representation theory

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Abstract

The consequences of replica-symmetry breaking on the structure of ultrametric matrices appearing in the theory of disordered systems is investigated with the help of representation theory, and the results are compared with those obtained by Temesvári, De Dominicis and Kondor.

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1 Introduction

The technique of replica-symmetry breaking provides a general framework to describe the microscopic properties of low-temperature disordered systems. Originally developed in the theory of spin glasses [1], this method had found applications in a wide variety of problems, such as the theory of random manifolds [2, 3, 4], vortex pinning [5], random-field problems [6, 7], etc. In these theories randomness is handled via the replica trick, and the multitude of equilibrium phases is captured by breaking the permutation symmetry between replicas.

In the replica method the free-energy $F = F(q_{\alpha\beta})$ depends on a set of order parameters $q_{\alpha\beta}$, where the replica indices α, β take integer values in the set $\{1, 2, \ldots, n\}$ and the order parameter matrix is symmetric with zero diagonal entries. The free-energy is independent of the labeling of the replicas, i.e. F is invariant under the transformations $q_{\alpha\beta} \to q_{\pi(\alpha)\pi(\beta)}$ for $\pi \in S_n$, where S_n denotes the symmetric group of rank n, that is the group of all permutations of the integers $\{1, 2, \ldots, n\}$.

Depending on the value of the parameters in F, the stationary points of the free-energy are either symmetrical, meaning that all of their off-diagonal components are equal, or replica-symmetry breaking. As usual, symmetry breaking means that the ground state is invariant only under a proper subgroup of the underlying S_n symmetry group of the theory. Many important features of the theory follow from the residual symmetry of the ground state by standard arguments based on the Wigner-Eckart theorem.

The successful Ansatz for the symmetry breaking pattern, first proposed by Parisi [1], looks as follows. Let R be a positive integer and let the positive

integers $p_0 = n, p_1, p_2, \ldots, p_R$ be such that p_{i+1} divides p_i. The $n \times n$ matrix $q_{\alpha\beta}$ is divided into blocks of size $p_1 \times p_1$, and a common value q_0 is assigned to all matrix elements outside the diagonal blocks. Next, the diagonal blocks are further divided into blocks of size $p_2 \times p_2$ and the value $q_1 \neq q_0$ is assigned to all elements inside the diagonal blocks of size $p_1 \times p_1$ but outside the diagonal blocks of size $p_2 \times p_2$, and so on down to the innermost blocks of size $p_R \times p_R$, where the matrix elements are q_R, except the diagonal of the whole matrix where they are all zero.

The residual symmetry group is by definition that subgroup of S_n which leaves the saddle-point invariant, i.e.

$$H_{(p_0,p_1,\ldots,p_R)} = \{\pi \in S_n |\ q_{\pi(\alpha)\pi(\beta)} = q_{\alpha\beta} \text{ for all } \alpha, \beta = 1, 2, \ldots, n\}. \quad (1)$$

The structure of this group for a Parisi-type saddle-point is captured by the notion of a wreath product of symmetric groups [8]. Let k be a divisor of n and divide the natural numbers $\{1, 2, \ldots, n\}$ into l $(= n/k)$ blocks of length k. The wreath product $S_k \wr S_l$ is the group of permutations which move blocks as a whole with permutations from S_l, and also the elements inside the blocks with permutations from S_k. Generalization to multiple wreath products is obvious and it can be shown that the wreath product is associative, i.e. brackets can be omitted. The residual symmetry group at a Parisi-type saddle-point, which we shall denote by H in the following, is isomorphic to the multiple wreath product

$$S_{p_R} \wr S_{p_{R-1}/p_R} \wr \ldots \wr S_{p_0/p_1}. \quad (2)$$

The second derivative $M_{\alpha\beta,\gamma\delta} := \partial^2 F / \partial q_{\alpha\beta} \partial q_{\gamma\delta}$ of the free energy evaluated at a Parisi-type saddle-point is a four-index quantity whose special properties following from the symmetry breaking pattern are usually referred to as ultrametricity. The characterization of a generic ultrametric matrix - block-diagonalization, spectral decomposition - was given in [10], where it was shown that there exists a basis such that the operator M is block-diagonal, containing only blocks of sizes $R + 1$ and 1. It was the desire to understand the group theoretic origin of this result that led us to the present representation theoretic study of ultrametricity. Clearly, the advantage of the group theoretic analysis is that it may be readily generalized to more complex situations, e.g. the study of higher rank ultrametric operators (i.e. higher derivatives of the free-energy), whose properties are of prime interest for a better understanding of the underlying physical theories.

2 Ultrametric matrices

Let us denote by $\mathcal{Q}$ the space of order parameters, i.e. the linear space of symmetric $n \times n$ matrices with zeros on the main diagonal. The free-energy is a real-valued function on $\mathcal{Q}$, invariant under the action $q_{\alpha\beta} \to q_{\pi(\alpha)\pi(\beta)}$ of the symmetric group S_n. The above action realizes a linear representation D of S_n on the space $\mathcal{Q}$, whose decomposition into irreducibles is given by $(n \geq 5)$ [11]

$$D = [n-2, 2] \oplus [n-1, 1] \oplus [n], \quad (3)$$

where we have used the usual labeling of the irreps of S_n via partitions of n. We see that only three irreducible components appear, which - by analogy with the representations of the general linear group - may be termed as the "tensor", "vector" and "scalar", respectively.

The Hessian $M = \partial^2 F/\partial q^2$ may be viewed as a linear operator $M : \mathcal{Q} \to \mathcal{Q}$. When evaluated at a Parisi-type saddle-point with residual symmetry group H, the invariance of F with respect to the action of S_n implies $D(\pi)MD^{-1}(\pi) = M$, in other words

$$[D(\pi), M] = 0 \tag{4}$$

for all $\pi \in H$. This commutation rule is the abstract algebraic expression of the ultrametricity of M, and the problem is to find out the implications of this property on the structure of the operator, e.g. the number of different eigenvalues together with their multiplicities.

Such conclusions may be drawn by a clever application of the Wigner-Eckart theorem. For suppose we know the decomposition into irreducibles of the restriction $D \downarrow H$ of the representation D to the residual subgroup H :

$$D \downarrow H = \bigoplus_i m_i C^{(i)}, \tag{5}$$

where the $C^{(i)}$ denote the irreps of H, and m_i is the multiplicity of the corresponding irrep. Then the Wigner-Eckart theorem tells us that in a suitable basis the ultrametric matrix M is block-diagonal, having blocks of size m_i appearing with multiplicity d_i, equal to the dimension of the irrep $C^{(i)}$. Moreover, the diagonal blocks may be written down explicitly by applying suitable projection operators completely determined by the irreps $C^{(i)}$.

3 The decomposition of $D \downarrow H$

The structure of the residual subgroup H and of its irreducible representations change markedly as we increase the number R of symmetry breaking steps. It is therefore natural to try to describe this process inductively, starting from the symmetric case where $R = 0$, and going on to the more complicated cases step-by-step.

3.1 The $R = 0$ case ($H = S_n$)

In this case there is no symmetry breaking. As we have seen previously, D can be decomposed into three irreps: $[n-2, 2] \oplus [n-1, 1] \oplus [n]$, which we'll denote in the sequel by t_0, v_0 and s_0 respectively, the subscript referring to the $R = 0$ case. An ultrametric operator M satisfying (4) has accordingly three different eigenvalues corresponding to the above irreps, with respective multiplicities $\frac{1}{2}n(n-3)$, $n-1$ and 1.

3.2 The $R = 1$ case ($H = S_k \wr S_l$, $kl = n$)

We need to find the irreducible constituents of $D \downarrow H$. The restriction of the identity representation is trivial, but that of the vector and tensor requires a more sophisticated analysis. For details of the representation theory of wreath products we refer to Appendix A. While the proof works only for $k, l \geq 5$, the result turns out to be valid for $k, l \geq 4$ as well.

To decompose into irreducibles the restriction to $S_k \wr S_l$ of an irrep of S_{kl}, one can apply the following simple procedure :

- One computes the restriction of the irrep to S_k by repeated application of the so-called branching law, which describes the decomposition of the restriction of any irrep of S_n to S_{n-1}.
- From Eqs. (28) and (29) of the Appendix one can compute the decomposition of any irrep of $S_k \wr S_l$ into irreps of S_k.
- By the transitivity of restriction, the above decompositions should agree, which constrains strongly the allowed irreducible constituents of the restriction to $S_k \wr S_l$.
- If there is still some ambiguity left in the decomposition, comparison of the character values at some specific elements will fix the result completely.

Applying the above procedure to the restriction $v_0 \downarrow H$ results in the decomposition

$$v_0 \downarrow = v_1 \oplus v_1', \tag{6}$$

where v_1 and v_1' are certain irreducible representations of $S_k \wr S_l$, to be defined in the Appendix. Here and from now on $\downarrow$ denotes the restriction to the next level, i.e. from $H_{(p_0,p_1,\ldots,p_i)}$ to the subgroup $H_{(p_0,p_1,\ldots,p_i,p_{i+1})}$. The analogous result for the tensor representation t_0 reads

$$t_0 \downarrow = t_1 \oplus v_1 \oplus v_1^2 \oplus v_1 v_1' \oplus t_1' \oplus v_1' \oplus s, \tag{7}$$

where t_1, v_1^2, $v_1 v_1'$ and t_1' denote again irreducible representations of $S_k \wr S_l$ to be defined in the Appendix. Putting all together, we get in this case the result

$$D \downarrow = t_1 \oplus 2v_1 \oplus v_1^2 \oplus v_1 v_1' \oplus t_1' \oplus 2v_1' \oplus 2s, \tag{8}$$

i.e. a total of seven different irreducible constituents, three of them with multiplicity 2. According to this, an ultrametric matrix has ten different eigenvalues, whose multiplicities are determined by the dimensions of the above irreps (cf. Table 1).

3.3 Generalization to $R > 1$

For a start, let's restrict the above mentioned $S_k \wr S_l$ irreps to $(S_p \wr S_q) \wr S_l$ ($pq = k$) and decompose them. The method together with some illustrating examples is described in Appendix B. The definitions of the irreps to appear

in this subsection are also to be found there. For the decomposition of $v_1 \downarrow$ we obtain

$$v_1 \downarrow = v_2 \oplus v'_2, \tag{9}$$

while for $t_1 \downarrow$ we have

$$t_1 \downarrow = t_2 \oplus v_2 \oplus v_2^2 \oplus v_2 v'_2 \oplus t'_2 \oplus v'_2 \oplus v'_1 \oplus s \tag{10}$$

where the subscript 2 refers again the $R = 2$. The $R = 1$ level representations which contain trivial base representations - i.e. t'_1, v'_1 and s - remain irreducible under restriction (so we denote the restricted irreps with the same symbols), while $v_1 v'_1$ restricts simply as

$$v_1 v'_1 \downarrow = v_2 v'_1 \oplus v'_2 v'_1. \tag{11}$$

The most tricky case is the decomposition of $v_1^2 \downarrow$. The result reads

$$v_1^2 \downarrow = v_2 \overset{1}{\bowtie} v_2 \oplus v_2 \overset{1}{\bowtie} v'_2 \oplus v'_2 \overset{1}{\bowtie} v'_2. \tag{12}$$

To make the $R = 2$ step clear, Figure 1. shows the decomposition tree of $t_0 \downarrow H$ up to this level.

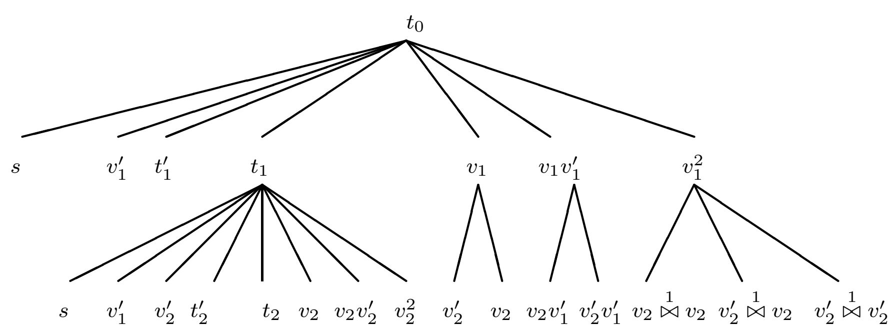

Figure 1: Decomposition of t_0

The generalization to higher R's can be described inductively. Until now to make the notation easier to understand we used p, q and l instead of $p_0 = n = pql$, $p_1 = pq$ and $p_2 = p$, but from now on we shall proceed with the p_i's ($i = 1, 2, \ldots, R$). The inductive definition of the representations to appear in what follows is to be found in the Appendix.

$v_i \downarrow$ splits into two representations, similarly to (6)

$$v_i \downarrow = v_{i+1} \oplus v'_{i+1}, \tag{13}$$

while $t_i \downarrow$ splits into $i + 7$ different representations:

$$t_i \downarrow = t_{i+1} \oplus v_{i+1} \oplus v_{i+1}^2 \oplus v_{i+1} v'_{i+1} \oplus t'_{i+1} \oplus v'_{i+1} \oplus v'_i \oplus \ldots \oplus v'_1 \oplus s. \tag{14}$$

As a general rule we can state that the inertia factor representations, denoted by a prime, never decompose further (so we can denote them with the same symbol), e.g. t'_i and v'_i remain irreducible under further restriction. $v_i v'_i$ restricts according to

$$v_i v'_i \downarrow = v_{i+1} v'_i \oplus v'_{i+1} v'_i. \tag{15}$$

The above decomposition generates representations of the form $v_i v'_j$ and $v'_i v'_j$ with $(i \geq j)$. The restriction rule for the first type reads

$$v_i v'_j \downarrow = v_{i+1} v'_j \oplus v'_{i+1} v'_j, \tag{16}$$

while the second type remains irreducible, containing only inertia factor representations. Once again the most complicated case is the decomposition of $v_i^2 \downarrow$

$$v_i^2 \downarrow = v_{i+1} \overset{i}{\bowtie} v_{i+1} \oplus v_{i+1} \overset{i}{\bowtie} v'_{i+1} \oplus v'_{i+1} \overset{i}{\bowtie} v'_{i+1} \tag{17}$$

which gives birth to three new class of representations.

$$v_{i+1} \overset{i}{\bowtie} v_{i+1} \downarrow = v_{i+2} \overset{i}{\bowtie} v_{i+2} \oplus v_{i+2} \overset{i}{\bowtie} v'_{i+2} \oplus v'_{i+2} \overset{i}{\bowtie} v'_{i+2}. \tag{18}$$

It is now obvious that after a certain number of steps starting with v_i^2 we obtain representations of the form

$$v_i \overset{j}{\bowtie} v_i \qquad v_i \overset{j}{\bowtie} v'_k \qquad v'_i \overset{j}{\bowtie} v'_k. \tag{19}$$

The third ones do not decompose further, the second ones split as

$$v_i \overset{j}{\bowtie} v'_i \downarrow = v_{i+1} \overset{j}{\bowtie} v'_i \oplus v'_{i+1} \overset{j}{\bowtie} v'_i \tag{20}$$

and the first ones split according to

$$v_i \overset{j}{\bowtie} v_i \downarrow = v_{i+1} \overset{j}{\bowtie} v_{i+1} \oplus v_{i+1} \overset{j}{\bowtie} v'_{i+1} \oplus v'_{i+1} \overset{j}{\bowtie} v'_{i+1}. \tag{21}$$

Finally, after having performed R reduction steps, we obtain the result summarized in Table 1 for the restriction of the representation D. The domain of the variables are $1 \leq i, j \leq R$ and $1 \leq k \leq R-1$. The classification of the irreps into families accords with that of [10].

4 Discussion

The irreps are divided into three families: L, A and R and the latter is subdivided into three subfamilies R_1, R_2 and R_3. The family L consists of the trivial irrep s, A consists of the "vector-like" irreps, and R includes the other irreps, which are characterized by the fact that they all originate from the tensor representation t_0. This classification accords that of [10].

What kind of conclusions may be drawn from the above decomposition about the structure of an arbitrary ultrametric matrix? As explained in section 2, the Wigner-Eckart theorem tells us that the matrix may be block-diagonalized in a suitable basis. To the irreps in the L and A families will

Family	*Symbol of irrep*	*Multiplicity*	*Dimension*
L	s	$R+1$	1
A	v_i'	$R+1$	$n(1/p_i - 1/p_{i-1})$
	v_R	$R+1$	$n(1 - 1/p_R)$
R_1	t_i'	1	$\frac{n}{2}(p_{i-1} - 3p_i)/p_i^2$
	t_R	1	$\frac{n}{2}(p_R - 3)$
R_2	$v_i'v_j'$, $(i > j)$	1	$n(p_{i-1} - 2p_i)/p_i(1/p_j - 1/p_{j-1})$
	v_Rv_i'	1	$n(p_{i-1} - 2p_i)/p_i(1 - 1/p_R)$
R_3	v_R^2	1	$\frac{n}{2}(p_{R-1} - p_R)(1 - 1/p_R)^2$
	$v_R \overset{k}{\bowtie} v_R$	1	$\frac{n}{2}(p_{j-1} - p_j)(1 - 1/p_R)^2$
	$v_i' \overset{k}{\bowtie} v_i'$, $(i > k)$	1	$\frac{n}{2}(p_{j-1} - p_j)(1/p_i - 1/p_{i-1})^2$
	$v_R \overset{k}{\bowtie} v_i'$, $(i > k)$	1	$n(p_{j-1} - p_j)(1 - 1/p_R)(1/p_i - 1/p_{i-1})$
	$v_i' \overset{k}{\bowtie} v_j'$, $(i > j > k)$	1	$n(p_{j-1} - p_j)(1/p_k - 1/p_{k-1})(1/p_i - 1/p_{i-1})$

Table 1: Irreducible constituents of $D \downarrow H$

correspond blocks of size $R+1$, with multiplicities equal to the dimension of the corresponding irreps, while the representations from the family R appear only once, i.e. to each of them is associated a single eigenvalue of the ultrametric matrix, whose multiplicity is again the dimension of the corresponding irrep. This is exactly the pattern found in [10] - without the use of group theory - for the spectral decomposition of an arbitrary ultrametric matrix.

In summary, we have seen that the structure of ultrametric matrices is to a large extent determined by the residual symmetry group, in complete accord with the results of [10]. While the primary goal of the present work was to elucidate the group theoretic background of that paper, it should be stressed that the results may be applied in further investigations of replica-symmetry breaking, e.g. in the analysis of the symmetry properties of the correlation functions. Besides this, they may lead to a better understanding of the symmetry structures present in the physically interesting limit $R \to \infty$, which is probably one of the most interesting features of the theory.

Acknowledgments

The application of representation theory techniques to the study of replica-symmetry breaking was pioneered by the late Claude Itzykson. We are grateful to I. Kondor and T. Temesvári for directing our attention to this field and for the many interesting discussions.

A Representations of wreath products

First of all, we sketch briefly the representation theory of wreath products $G \wr S_l$ for a finite permutation group G of degree k [11], which is a classical application of Clifford's theorem [12]. Let's divide the natural numbers $\{1, 2, \dots, n\}$ into blocks of length k and let $G^{(i)}$ denote the subgroup of S_n which permutes the numbers inside the i-th block ($i = 1, 2, \dots, l$). Clearly $G^{(i)} \cong G$. To obtain the irreps of the wreath product $G \wr S_l$ we follow the procedure outlined here:

- Let's first construct the so-called base group (containing no permutations moving whole blocks)

$$G^* = G^{(1)} \times G^{(2)} \times \ldots \times G^{(l)}. \tag{22}$$

  The irreps of this group are of the form

$$F^{(1)} \sharp F^{(2)} \sharp \ldots \sharp F^{(l)}, \tag{23}$$

  where $F^{(i)}$ is an irrep of $G^{(i)}$ and the symbol $\sharp$ denotes the outer tensor product. Let $D_1, D_2, \dots, D_r$ be all the irreps of G and define an l-partition $\Lambda = \langle \lambda_1, \lambda_2, \dots, \lambda_r \rangle$ which denote the situation where λ_1 of the $F^{(i)}$'s are equal to D_1 and λ_2 of them are equal to D_2, etc. Λ is called the type of the base group representation and it has the property $\sum_{i=1}^{r} \lambda_i = l$.

- Let's define the inertia factor:

$$S_\Lambda = \{\pi \in S_l \mid F^{(\pi(i))} = F^{(i)} \text{ for all } i = 1, 2, \dots, l\ \}, \tag{24}$$

  which is isomorphic with $\times_{i=1}^{r} S_{\lambda_i}$.

- Now we extend the representation from the base group to the $G \wr S_\Lambda$ inertia group:

$$\begin{aligned} &\overline{(F^{(1)}_{\alpha_1\beta_1} F^{(2)}_{\alpha_2\beta_2} \ldots F^{(i)}_{\alpha_l\beta_l}})(g_1, g_2, \dots, g_l; \sigma) = \\ &F^{(1)}_{\alpha_1\beta_{\sigma(1)}}(g_1) F^{(2)}_{\alpha_2\beta_{\sigma(2)}}(g_2) \ldots F^{(l)}_{\alpha_l\beta_{\sigma(l)}}(g_l) \end{aligned} \tag{25}$$

  for all $g_i \in G^{(i)}$ and $\sigma \in S_\Lambda$.

- Finally the general form of an irrep $\mathcal{D}$ of $G \wr S_l$ is the following:

$$(\overline{F^{(1)} \sharp F^{(2)} \sharp \ldots \sharp F^{(l)}} \otimes K) \uparrow G \wr S_l. \tag{26}$$

  Here K is an irrep of the inertia factor S_Λ and hence a tensor product of irreps of S_{λ_i}, i.e. $K = K_1 \sharp K_2 \sharp \ldots \sharp K_r$. An alternative – shorter – notation of (26) is

$$\langle D_1, K_1 \rangle \sharp \langle D_2, K_2 \rangle \sharp \ldots \sharp \langle D_r, K_r \rangle. \tag{27}$$

Let's consider the behaviour of the irrep $\mathcal{D}$ under restriction. Let's single out one of the factors of G^*, e.g. the first one, and consider the restriction $\mathcal{D} \downarrow G$. It follows from the construction that its decomposition into irreps of G is given by

$$\mathcal{D} \downarrow G = \bigoplus_{j=1}^{r} m_j D_j, \tag{28}$$

where the multiplicities m_j are given by

$$m_j = \frac{\lambda_j}{l \dim(D_j)} \dim(\mathcal{D}). \tag{29}$$

For the sake of definiteness we give the dimension of the whole wreath product irrep $\mathcal{D}$:

$$\dim(\mathcal{D}) = l! \prod_{i=1}^{r} \frac{\dim(K_i) \dim(D_i)^{\lambda_i}}{\lambda_i!} \tag{30}$$

B Decomposition in the $R = 1$ case

We shall illustrate the procedure outlined in 3.2 on the decomposition of the vector representation $[n-1,1] \downarrow (S_k \wr S_l)$. The branching law tells us that

$$[n-1,1] \downarrow S_k = [k-1,1] \oplus k(l-1)\ [k]. \tag{31}$$

Now taking into account the restriction rule (28) and (29) we conclude that the involved wreath product irreps may contain only vector and scalar irreps in the base representation, i.e. we may deal with a representation of the form $\langle v_0, K_1\rangle \sharp \langle s, K_2\rangle$. Since we have exactly one vector irrep in (31), hence there must be a wreath product irrep constituent of the decomposition with $\lambda_1 = 1$ and $\dim(K_2) = 1$ ($\dim(K_2)$ denotes the dimension of the irrep K_2). This implies $K_1 = [1]$ and since the only one-dimensional irreps are the trivial and the alternating: $K_2 = [l-1]$ or $K_2 = [1^l]$. Furthermore there must be another constituent (or other constituents) which contain no vector irrep factor in the base representation, i.e. with $\lambda_1 = 0$ and $\lambda_2 = l$. The dimension of the original v_0 is $(kl-1)$ so the remaining dimension is $(l-1)$ which can be filled in several ways: we can choose $K_2 = [l-1,1]$ or we can choose $l-1$ one-dimensional irreps with $K_2 = [l]$ or $K_2 = [1^l]$.

It is possible to further reduce the number of the possibilities using the characters of the representations. Evaluating the characters of both the original and the candidate representation on the elements (1 2) and (1 2 3) (permutating the blocks) we end up with only one remaining version:

$$v_0 \downarrow = \langle s, [l-1,1]\rangle \oplus \langle v_0, [1]\rangle \sharp \langle s, [l-1]\rangle. \tag{32}$$

We proved that the restriction of v_0 to $S_k \wr S_l$ can be decomposed according to (32). This decomposition holds for $k, l \geq 2$. To simplify the notation let's introduce the symbol v_1' for the first part of the decomposition and v_1 for the second. The subscript 1 at both symbol denotes the $R = 1$ case.

In case of the decomposition of $t_0\downarrow$ we just briefly sketch the definitions of the resulting constituents. Since here the branching law results [k-2,2] irreps too, the base group representations will contain scalar, vector and tensor as well:

$$\begin{aligned} t_1 &= \langle t_0, [1]\rangle\sharp\langle s, [l-1]\rangle \\ v_1v_1' &= \langle v_0, [1]\rangle\sharp\langle s, [l-1,1]\rangle \\ v_1^2 &= \langle v_0, [2]\rangle\sharp\langle s, [l-2]\rangle \\ t_1' &= \langle s, [l-2,2]\rangle \end{aligned} \tag{33}$$

C Generalization to $R > 1$

Let's consider v_1 as an example:

$$\begin{aligned} &([k-1,1]_1\sharp[k]_2\sharp\ldots\sharp[k]_l \otimes [l-1]\uparrow S_k\wr S_l)\downarrow S_p\wr S_q\wr S_l = \\ &[k-1,1]_1\downarrow(S_p\wr S_q)\sharp[k]_2\sharp\ldots\sharp[k]_l\otimes[l-1]\uparrow S_p\wr S_q\wr S_l \quad . \end{aligned} \tag{34}$$

Here we could omit the overline above the base irrep since it has no effect. The decomposition of $[k-1,1]\downarrow S_p\wr S_q$ is already known, so making use of the distributivity of the tensor product we obtain the following constituents:

$$\begin{aligned} v_2 &= \langle v_1, [1]\rangle\sharp\langle s, [l-1]\rangle, \\ v_2' &= \langle v_1', [1]\rangle\sharp\langle s, [l-1]\rangle \end{aligned} \tag{35}$$

We have simply changed the $[k-1,1]$ factor to two different $S_p\wr S_q$ irreps (v_1 and v_1'). Luckily the resulting representations are irreducible not like at the decomposition of $t_1\downarrow$; when we change the $[k-2,2]$ factor to the trivial representation of $S_p\wr S_q$ and take a look at the result

$$(\text{trivials } \uparrow S_p\wr S_q)\sharp \text{ trivials } \otimes[1][l-1]\uparrow S_p\wr S_q\wr S_l \tag{36}$$

we notice the the base representation is the identity so the inertia factor should be the full S_l and not $S_1\times S_{l-1}$ as considered. Thus we have to find a decomposition for the $[1]\sharp[l-1]\uparrow S_l$ representation of the inertia factor to irreps of S_l. It's dimension is l so it may consist of l copies of one-dimensional irreps or one vector-dimensional and one one-dimensional irrep. The decision is made again by evaluating characters on the two particular elements (1 2) and (1 2 3). Finally we have: $s_2 = v_1'\oplus s$. The definitions of $v_2v_1'\oplus v_2'v_1'$ are

$$\begin{aligned} v_2v_1' &= \langle v_1, [1]\rangle\sharp\langle s, [l-1,1]\rangle, \\ v_2'v_1' &= \langle v_1', [1]\rangle\sharp\langle s, [l-1,1]\rangle. \end{aligned} \tag{37}$$

At the $R=2$ level $v_1^2\downarrow$ is the only representation where we cannot use the above mentioned method: there are two non-trivial factors in the base

representation so we cannot omit the overline and use the distributivity of the tensor product. So we apply the procedure similar to the one used at the $R = 1$ case and obtain the result

$$\begin{aligned} v_2 \overset{1}{\bowtie} v_2 &= \langle v_1, [2] \rangle \sharp \langle s, [l-2] \rangle, \\ v_2 \overset{1}{\bowtie} v_2' &= \langle v_1, [1] \rangle \sharp \langle v_1', [1] \rangle \sharp \langle s, [l-2] \rangle, \\ v_2' \overset{1}{\bowtie} v_2' &= \langle v_1', [2] \rangle \sharp \langle s, [l-2] \rangle. \end{aligned} \tag{38}$$

To conclude, let's give the precise definition of the representations relevant to our work. To do this, we shall define inductively certain irreps of the multiple wreath-product $S_{n_0} \wr S_{n_1} \wr \ldots \wr S_{n_R}$. We define the representations s_0, v_0 and t_0 of S_{n_0} as

$$s_0 = [n_0] \qquad v_0 = [n_0 - 1, 1] \qquad t_0 = [n_0 - 2, 2] \tag{39}$$

We then define irreps s_i, v_i and t_i of $S_{n_0} \wr \ldots S_{n_i}$ via the inductive rule

$$\begin{aligned} s_{i+1} &= \langle s_i, [n_{i+1}] \rangle \\ v_{i+1} &= \langle v_i, [1] \rangle \sharp \langle s_i, [n_{i+1} - 1] \rangle \\ t_{i+1} &= \langle t_i, [1] \rangle \sharp \langle s_i, [n_{i+1} - 1] \rangle \end{aligned} \tag{40}$$

Note that s_i is just the trivial representation for all i, so we can safely omit the subscript and refer to it simply as s.

We also need some other types of irreps, which may be constructed starting from the representations $v_1', t_1', v_1 v_1'$ and v_1^2 of $S_{n_0} \wr S_{n_1}$ defined as

$$\begin{aligned} v_1' &= \langle s_0, [n_1 - 1, 1] \rangle \\ t_1' &= \langle s_0, [n_1 - 2, 2] \rangle \\ v_1 v_1' &= \langle v_0, [1] \rangle \sharp \langle s_0, [n_1 - 1, 2] \rangle \\ v_1^2 &= \langle v_0, [2] \rangle \sharp \langle s_0, [n_1 - 2] \rangle \end{aligned} \tag{41}$$

The inductive step then reads

$$\begin{aligned} v_{i+1}' &= \langle v_i', [1] \rangle \sharp \langle s, [n_{i+1} - 1] \rangle \\ t_{i+1}' &= \langle t_i', [1] \rangle \sharp \langle s, [n_{i+1} - 1] \rangle \\ v_{i+1} v_1' &= \langle v_i, [1] \rangle \sharp \langle s, [n_{i+1} - 2, 1] \rangle \\ v_{i+1} v_{j+1}' &= \langle v_i v_j', [1] \rangle \sharp \langle s, [n_{i+1} - 1] \rangle \qquad j < i \\ v_{i+1}' v_1' &= \langle v_i', [1] \rangle \sharp \langle s, [n_{i+1} - 2, 1] \rangle \end{aligned}$$

$$
\begin{aligned}
v'_{i+1}v'_{j+1} &= \langle v'_i v'_j, [1]\rangle \sharp \langle s, [n_{i+1}-1]\rangle \qquad\qquad j<i \\
v^2_{i+1} &= \langle v_i^2, [1]\rangle \sharp \langle s, [n_{i+1}-1]\rangle
\end{aligned} \tag{42}
$$

We can now define inductively all the remaining representations that we need $(i > j > k)$:

$$
\begin{aligned}
v_{i+1} \overset{1}{\bowtie} v_{i+1} &= \langle v_i, [2]\rangle \sharp \langle s, [n_{i+1}-2]\rangle \\
v_{i+1} \overset{k+1}{\bowtie} v_{i+1} &= \langle v_i \overset{k}{\bowtie} v_i, [1]\rangle \sharp \langle s, [n_{i+1}-1]\rangle \\
v_{i+1} \overset{1}{\bowtie} v'_{j+1} &= \langle v_i, [1]\rangle \sharp \langle v'_j, [1]\rangle \sharp \langle s, [n_{i+1}-2]\rangle \\
v_{i+1} \overset{k+1}{\bowtie} v'_{j+1} &= \langle v_i \overset{k}{\bowtie} v'_j, [1]\rangle \sharp \langle s, [n_{i+1}-1]\rangle \\
v'_{i+1} \overset{1}{\bowtie} v'_{j+1} &= \langle v'_i, [1]\rangle \sharp \langle v'_j, [1]\rangle \sharp \langle s, [n_{i+1}-2]\rangle \\
v'_{i+1} \overset{k+1}{\bowtie} v'_{j+1} &= \langle v'_i \overset{k}{\bowtie} v'_j, [1]\rangle \sharp \langle s, [n_{i+1}-1]\rangle
\end{aligned} \tag{43}
$$

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